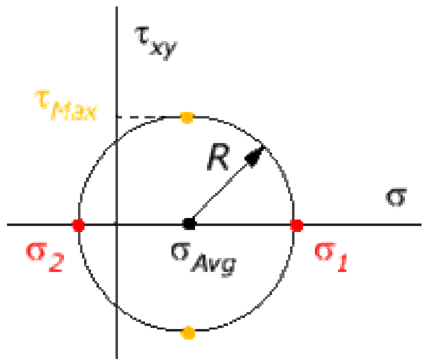
Solid Mechanics: Stress
Mohr's Circle for Plane Stress

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,



The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stesses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the principal (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress transformation formulas for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

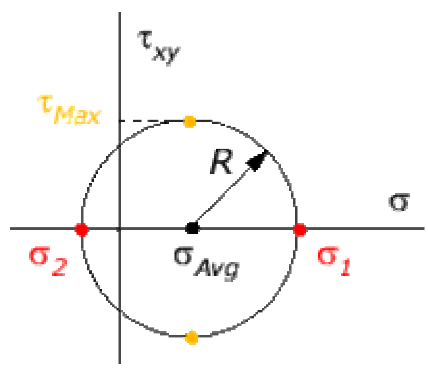
This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two principal stresses, and τ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg}, and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{\mathrm{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{\mathrm{Avg}}\right)^2 + \tau_{x'y'}{}^2 = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,



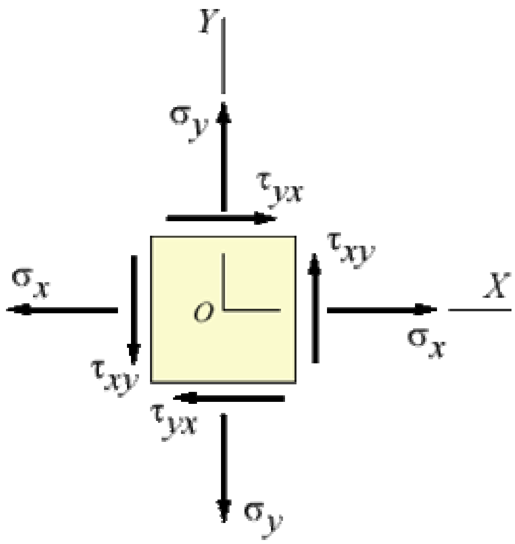
Related Topics

The procedure of drawing a Mohr's Circle from a given stress state is discussed in the Mohr's Circle usage page.

The Mohr's Circle for plane strain can also be obtained from similar procedures.

Solid Mechanics: Stress
Plane Stress and Coordinate Transformations

Plane State of Stress



A class of common engineering problems involving stresses in a thin plate or on the free surface of a structural element, such as the surfaces of thin-walled pressure vessels under external or internal pressure, the free surfaces of shafts in torsion and beams under transverse load, have one principal stress that is much smaller than the other two. By assuming that this small principal stress is zero, the three-dimensional stress state can be reduced to two dimensions. Since the remaining two principal stresses lie in a plane, these simplified 2D problems are called **plane stress** problems.

Assume that the negligible principal stress is oriented in the *z*-direction. To reduce the 3D stress matrix to the 2D plane stress matrix, remove all components with *z* subscripts to get,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$ for static equilibrium. The sign convention for positive stress components in plane stress is illustrated in the above figure on the 2D element.

Coordinate Transformations

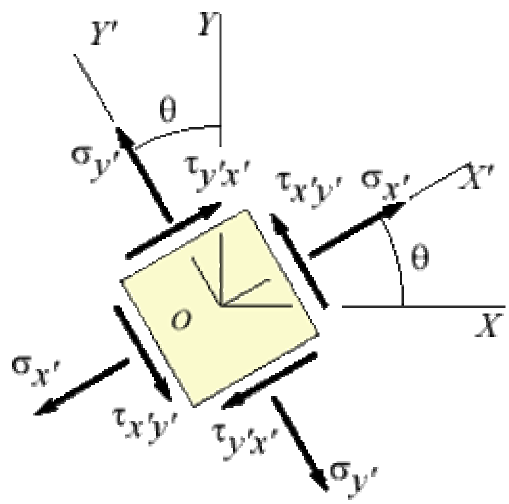
The coordinate directions chosen to analyze a structure are usually based on the shape of the structure. As a result, the direct and shear stress components are associated with these directions. For example, to analyze a bar one almost always directs one of the coordinate directions along the bar's axis.

Nonetheless, stresses in directions that do not line up with the original coordinate set are also important. For example, the failure plane of a brittle shaft under torsion is often at a 45° angle with respect to the shaft's axis. Stress transformation formulas are required to analyze these stresses.

The transformation of stresses with respect to the {x,y,z} coordinates to the stresses with respect to {x',y',z'} is performed via the equations,

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \quad\;\; = \sigma_x + \sigma_y - \sigma_{x'} \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

where θ is the rotation angle between the two coordinate sets (positive in the counterclockwise direction). This angle along with the stresses for the {x',y',z'} coordinates are shown in the figure below,



Solid Mechanics: Stress
Principal Stress for the Case of Plane Stress

Principal Directions, Principal Stress

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) and the shear stress ($\tau_{x'y'}$) vary smoothly with respect to the rotation angle θ, in accordance with the coordinate transformation equations. There exist a couple of particular angles where the stresses take on special values.

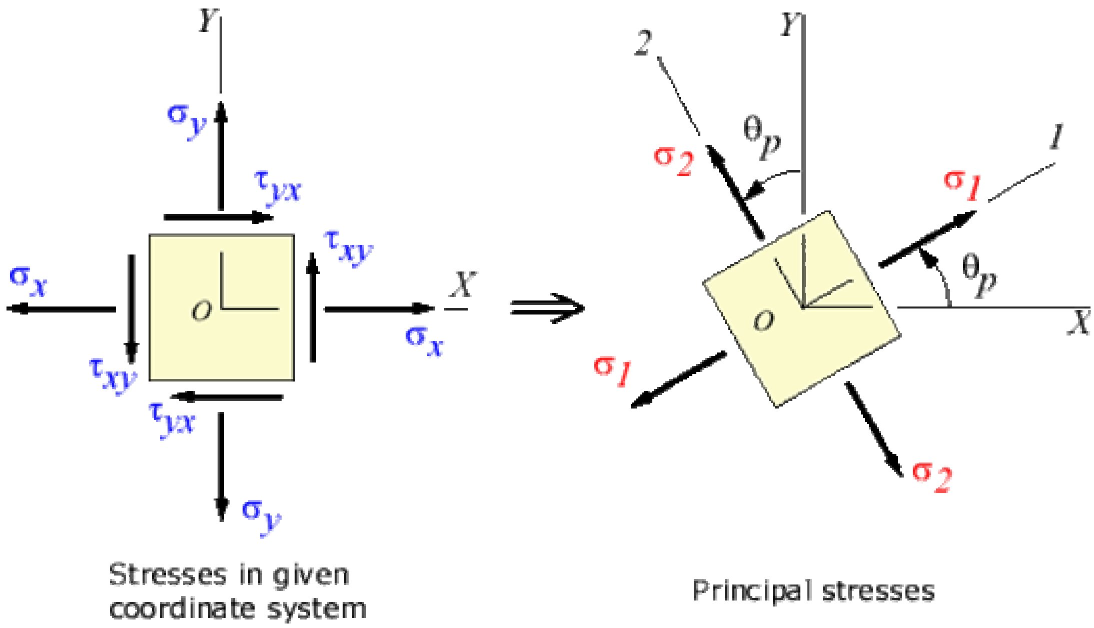
First, there exists an angle θ_p where the shear stress $\tau_{x'y'}$ becomes zero. That angle is found by setting $\tau_{x'y'}$ to zero in the above shear transformation equation and solving for θ (set equal to θ_p). The result is,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The angle θ_p defines the *principal directions* where the only stresses are normal stresses. These stresses are called *principal stresses* and are found from the original stresses (expressed in the *x*,*y*,*z* directions) via,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The transformation to the principal directions can be illustrated as:



Maximum Shear Stress Direction

Another important angle, θ_s, is where the maximum shear stress occurs. This is found by finding the maximum of the shear stress transformation equation, and solving for θ. The result is,

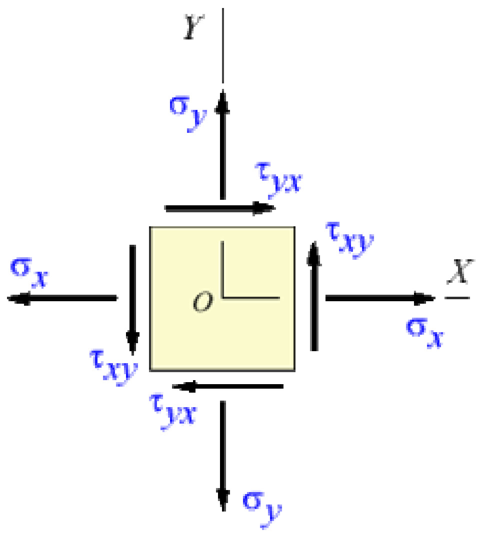
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

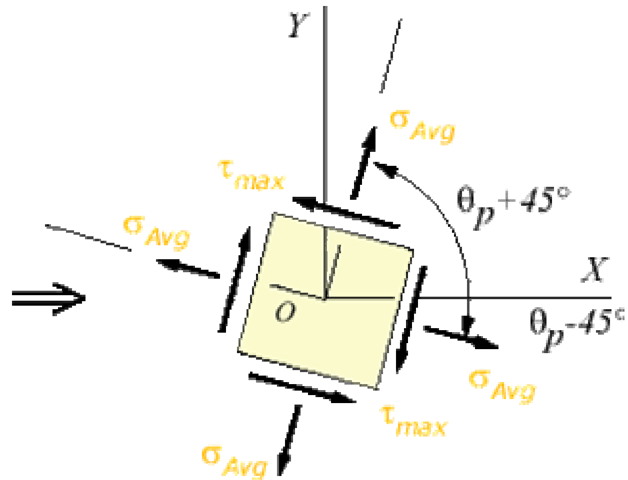
The maximum shear stress is equal to one-half the difference between the two principal stresses,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

The transformation to the maximum shear stress direction can be illustrated as:



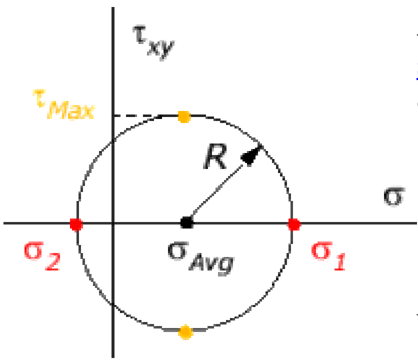
Stresses in given coordinate system



Maximum shear stress

Solid Mechanics: Stress
Mohr's Circle Usage in Plane Stress

Principal Stresses from Mohr's Circle



A chief benefit of Mohr's circle is that the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} are obtained immediately after drawing the circle,

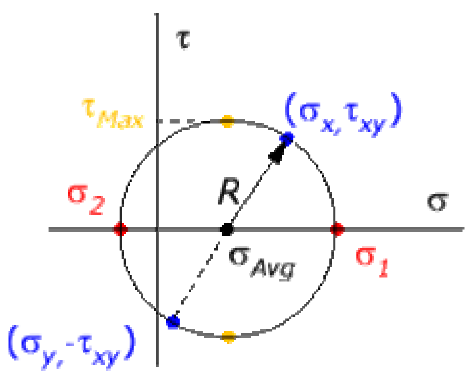
$$\begin{cases} \sigma_{1,2} = \sigma_{\text{Avg}} \pm R \\ \tau_{\text{Max}} = R \end{cases}$$

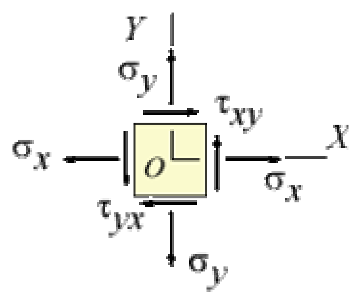
where,

$$\sigma_{\text{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Directions from Mohr's Circle

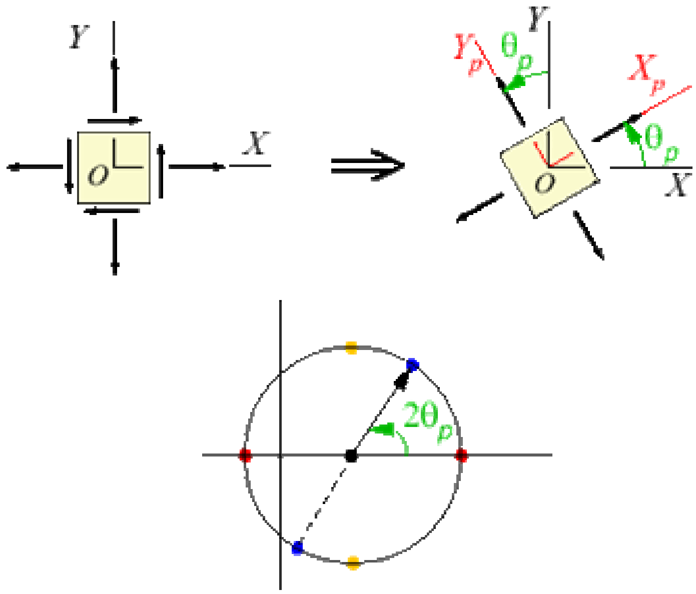
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a given point *O* in the body. They are expressed relative to the coordinates *XY*, as shown in the stress element at right below.





The Mohr's Circle for this general stress state is shown at left above. Note that it's centered at σ_{avg} and has a radius *R*, and that the two points $\{\sigma_x, \tau_{xy}\}$ and $\{\sigma_y, -\tau_{xy}\}$ lie on opposites sides of the circle. The line connecting σ_x and σ_y will be defined as L_{xy}.

The **angle** between the current axes (*X* and *Y*) and the **principal axes** is defined as θ_p, and is equal to one half the angle between the line L_{xy} and the σ-axis as shown in the schematic below,



A set of six Mohr's Circles representing most stress state possibilities are presented on the examples page.

Rotation Angle on Mohr's Circle

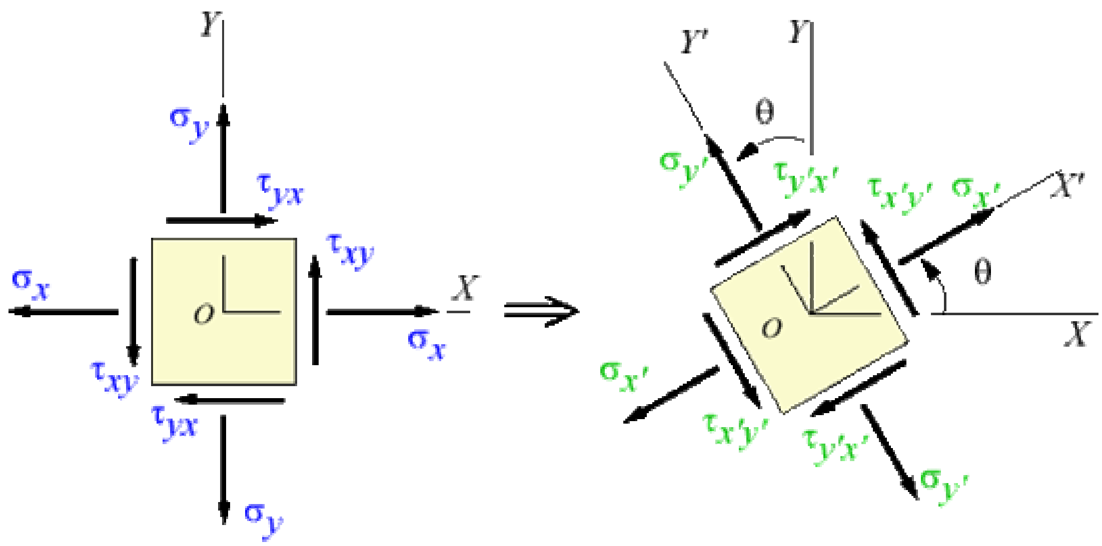
Note that the coordinate rotation angle θ_p is defined positive when starting at the *XY* coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal stress line (i.e. the σ-axis) and proceeding to the *XY* stress line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the *XY* coordinates, and on the other it starts on the principal coordinates.

Some books avoid this dichotomy between θ_p on Mohr's Circle and θ_p on the stress element by locating (σ_x, $-\tau_{xy}$) instead of (σ_x, τ_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Stress Transform by Mohr's Circle

Mohr's Circle can be used to transform stresses from one coordinate set to another, similar that that described on the plane stress page.

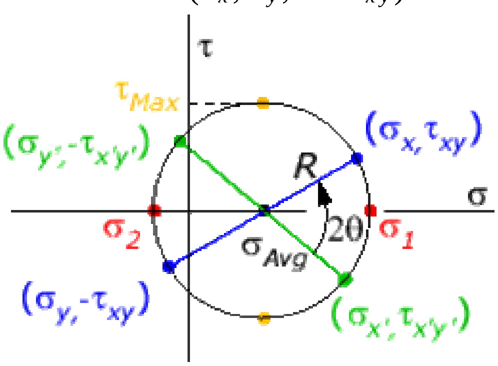
Suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a point *O* in the body, expressed with respect to the coordinates *XY*. We wish to find the stresses expressed in the new coordinate set *X'Y'*, rotated an angle θ from *XY*, as shown below:



Stresses at given coordinate system Stresses transformed to another coordinate

To do this we proceed as follows:

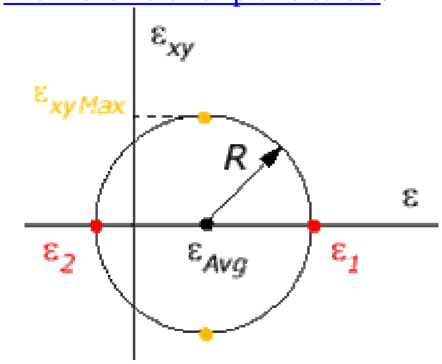
- Draw Mohr's circle for the **given stress state** (σ_x, σ_y, and τ_{xy}; shown below).
- Draw the line L_{xy} across the circle from (σ_x, τ_{xy}) to (σ_y, $-\tau_{xy}$).
- Rotate the line L_{xy} by 2*θ (twice as much as the angle between *XY* and *X'Y'*) and in the *opposite* direction of θ.
- The **stresses in the new coordinates** ($\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$) are then read off the circle.



Solid Mechanics: Strain
Mohr's Circle for Plane Strain

Mohr's Circle

Strains at a point in the body can be illustrated by Mohr's Circle. The idea and procedures are exactly the same as for Mohr's Circle for plane stress.



The two principal strains are shown in **red**, and the maximum shear strain is shown in **orange**. Recall that the normal strains are equal to the principal strains when the element is aligned with the principal directions, and the shear strain is equal to the maximum shear strain when the element is rotated 45° away from the principal directions.

As the element is rotated away from the principal (or maximum strain) directions, the normal and shear strain components will always lie on Mohr's Circle.

Derivation of Mohr's Circle

To establish the Mohr's circle, we first recall the strain transformation formulas for plane strain,

$$\begin{cases} \varepsilon_{x'} - \dfrac{\varepsilon_x + \varepsilon_y}{2} = \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \varepsilon_{xy}\cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the above two formulas we have,

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \varepsilon_{x'y'}{}^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2$$

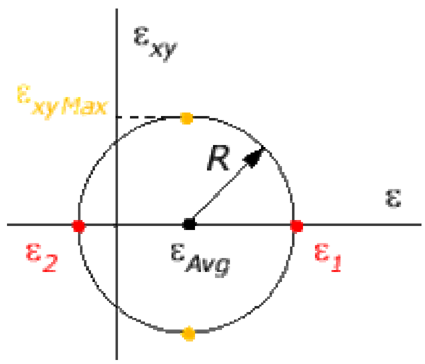
This equation is an equation for a circle. To make this more apparent, we can rewrite it as,

$$\left(\varepsilon_{x'} - \varepsilon_{\mathrm{Avg}}\right)^2 + \varepsilon_{x'y'}{}^2 = R^2$$

where,

$$\varepsilon_{\mathrm{Avg}} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

The circle is centered at the average strain value ε_{Avg}, and has a radius R equal to the maximum shear strain, as shown in the figure below,



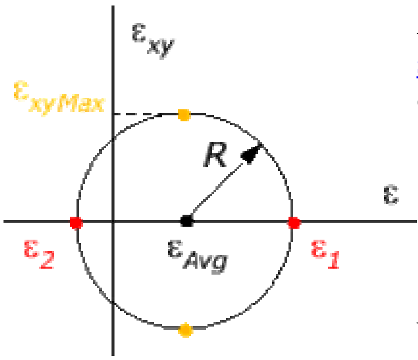
Related Topics

The procedure of drawing Mohr's Circle from a given strain state is discussed in the Mohr's Circle usage and examples pages.

The Mohr's Circle for plane stress can also be obtained from similar procedures.

Solid Mechanics: Strain
Mohr's Circle Usage in Plane Strain

Principal Strains from Mohr's Circle



A chief benefit of Mohr's circle is that the principal strains ε_1 and ε_2 and the maximum shear strain ε_{xyMax} are obtained immediately after drawing the circle,

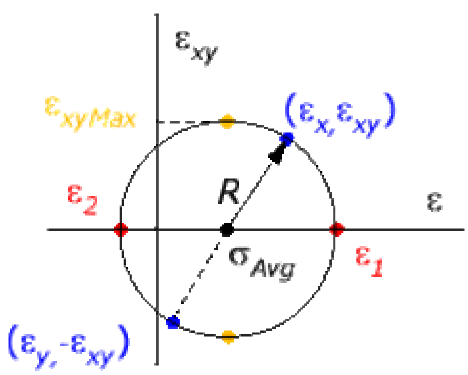
$$\begin{cases} \varepsilon_{1,2} = \varepsilon_{\text{Avg}} \pm R \\ \varepsilon_{xy_{\text{Max}}} = R \end{cases}$$

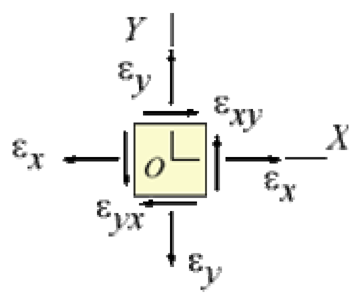
where,

$$\varepsilon_{\text{Avg}} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

Principal Directions from Mohr's Circle

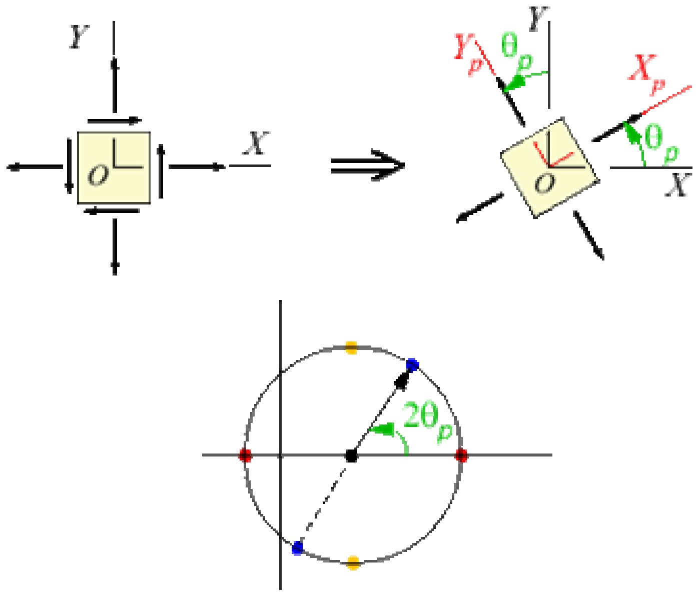
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear strains, ε_x, ε_y, and ε_{xy}, are obtained at a given point *O* in the body. They are expressed relative to the coordinates *XY*, as shown in the strain element at right below.





The Mohr's Circle for this general strain state is shown at left above. Note that it's centered at ε_{Avg} and has a radius *R*, and that the two points (ε_x, ε_{xy}) and (ε_y, -ε_{xy}) lie on opposites sides of the circle. The line connecting ε_x and ε_y will be defined as L_{xy}.

The **angle** between the current axes (*X* and *Y*) and the **principal axes** is defined as θ_p, and is equal to one half the angle between the line L_{xy} and the ε-axis as shown in the schematic below,



A set of six Mohr's Circles representing most strain state possibilities are presented on the examples page.

Rotation Angle on Mohr's Circle

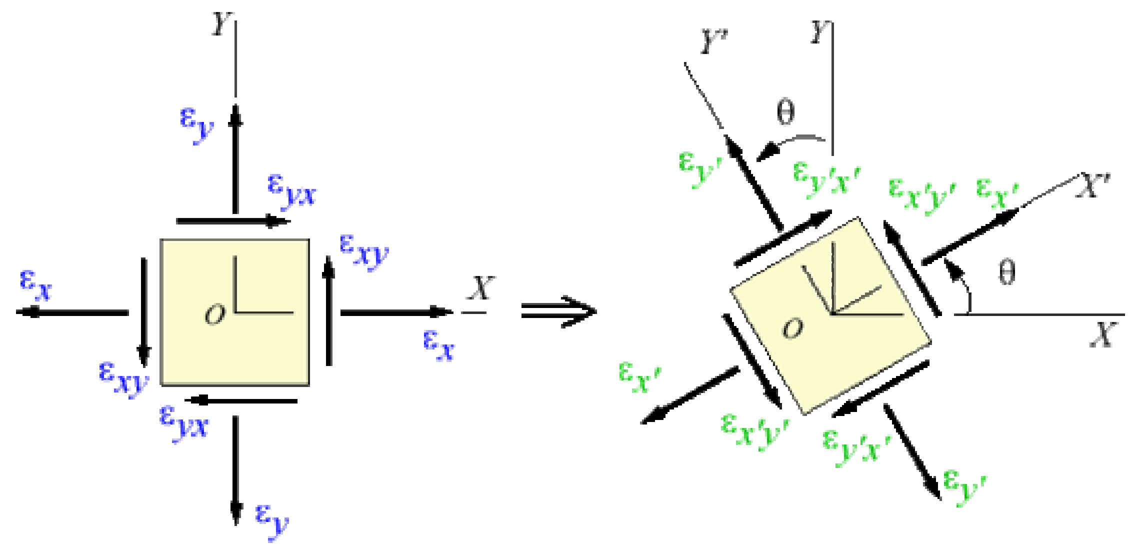
Note that the coordinate rotation angle θ_p is defined positive when starting at the *XY* coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal strain line (i.e. the ε-axis) and proceeding to the *XY* strain line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the *XY* coordinates, and on the other it starts on the principal coordinates.

Some books avoid the sign difference between θ_p on Mohr's Circle and θ_p on the stress element by locating (ε_x, $-\varepsilon_{xy}$) instead of (ε_x, ε_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Strain Transform by Mohr's Circle

Mohr's Circle can be used to transform strains from one coordinate set to another, similar that that described on the plane strain page.

Suppose that the normal and shear strains, ε_x, ε_y, and ε_{xy}, are obtained at a point *O* in the body, expressed with respect to the coordinates *XY*. We wish to find the strains expressed in the new coordinate set *X'Y'*, rotated an angle θ from *XY*, as shown below:

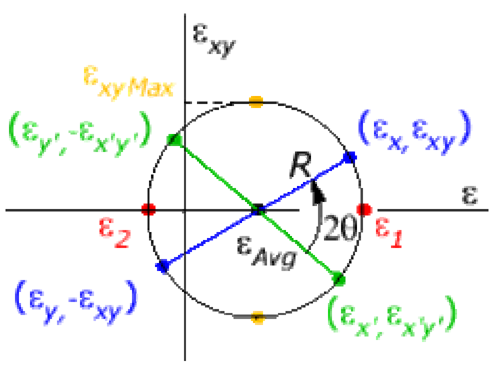


Strains at the given coordinate

Strains transformed to another coordinate

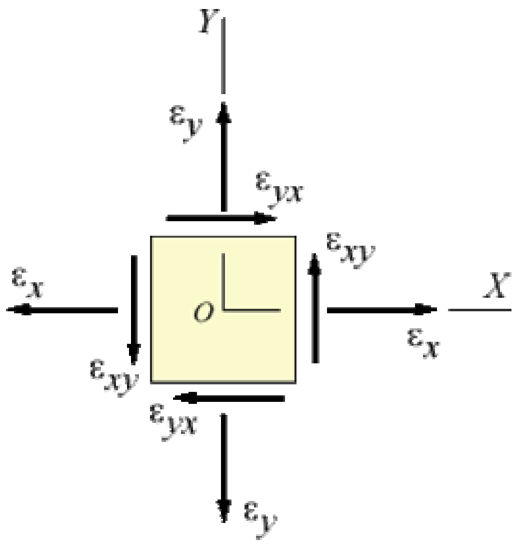
To do this we proceed as follows:

- Draw Mohr's circle for the **given strain state** (ε_x, ε_y, and ε_{xy}; shown below).
- Draw the line L_{xy} across the circle from (ε_x, ε_{xy}) to (ε_y, -ε_{xy}).
- Rotate the line L_{xy} by 2*θ (twice as much as the angle between *XY* and *X'Y'*) and in the *opposite* direction of θ.
- The **strains in the new coordinates** ($\varepsilon_{x'}$, $\varepsilon_{y'}$, and $\varepsilon_{x'y'}$) are then read off the circle.



Solid Mechanics: Strain
Plane Strain and Coordinate Transformations

Plane State of Strain



Some common engineering problems such as a dam subjected to water loading, a tunnel under external pressure, a pipe under internal pressure, and a cylindrical roller bearing compressed by force in a diametral plane, have significant strain only in a plane; that is, the strain in one direction is much less than the strain in the two other orthogonal directions. If small enough, the smallest strain can be ignored and the part is said to experience **plane strain**.

Assume that the negligible strain is oriented in the *z*-direction. To reduce the 3D strain matrix to the 2D plane stress matrix, remove all components with *z* subscripts to get,

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$$

where $\varepsilon_{xy} = \varepsilon_{yx}$ by definition.

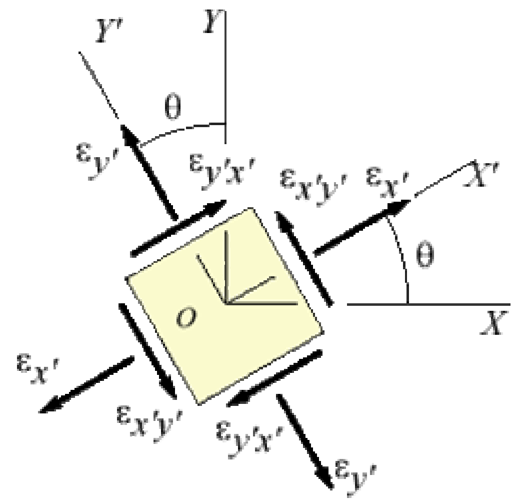
The sign convention here is consistent with the sign convention used in plane stress analysis.

Coordinate Transformation

The transformation of strains with respect to the $\{x,y,z\}$ coordinates to the strains with respect to $\{x',y',z'\}$ is performed via the equations,

$$\begin{cases} \varepsilon_{x'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} + \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta \\ \varepsilon_{y'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} - \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \varepsilon_{xy}\sin 2\theta \\ \quad = \varepsilon_x + \varepsilon_y - \varepsilon_{x'} \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \varepsilon_{xy}\cos 2\theta \end{cases}$$

The rotation between the two coordinate sets is shown here,



where θ is defined positive in the counterclockwise direction.

Solid Mechanics: Strain
Principal Strain for the Case of Plane Strain

Principal Directions, Principal Strain

The normal strains ($\varepsilon_{x'}$ and $\varepsilon_{y'}$) and the shear strain ($\varepsilon_{x'y'}$) vary smoothly with respect to the rotation angle θ, in accordance with the transformation equations given above. There exist a couple of particular angles where the strains take on special values.

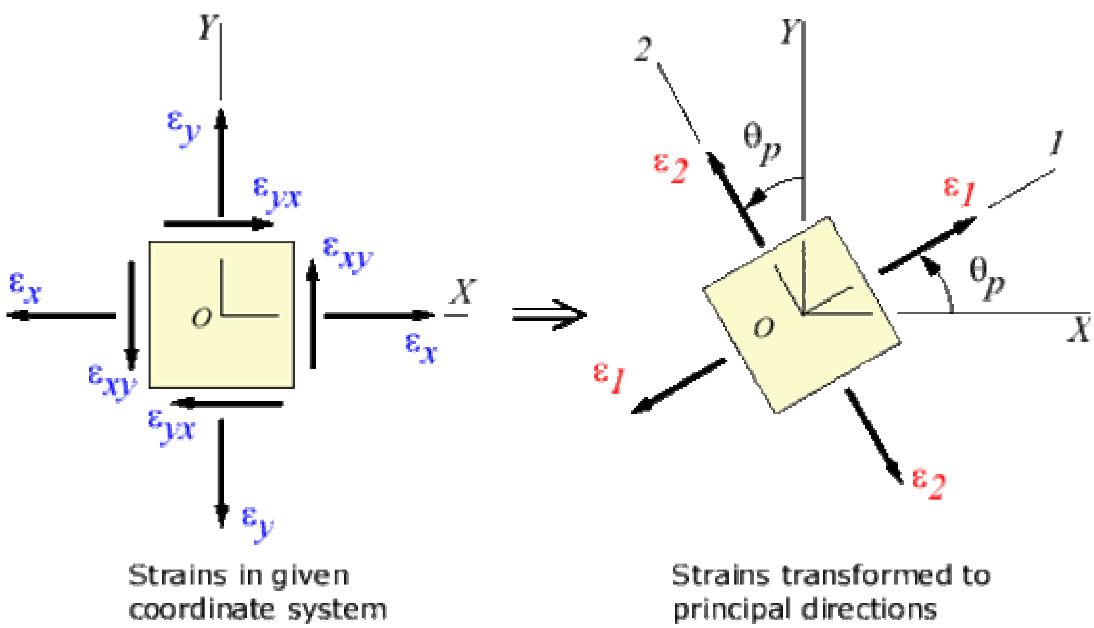
First, there exists an angle θ_p where the shear strain $\varepsilon_{x'y'}$ vanishes. That angle is given by,

$$\tan 2\theta_p = \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y}$$

This angle defines the *principal directions*. The associated *principal strains* are given by,

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

The transformation to the principal directions with their principal strains can be illustrated as:



Maximum Shear Strain Direction

Another important angle, θ_s, is where the maximum shear strain occurs and is given by,

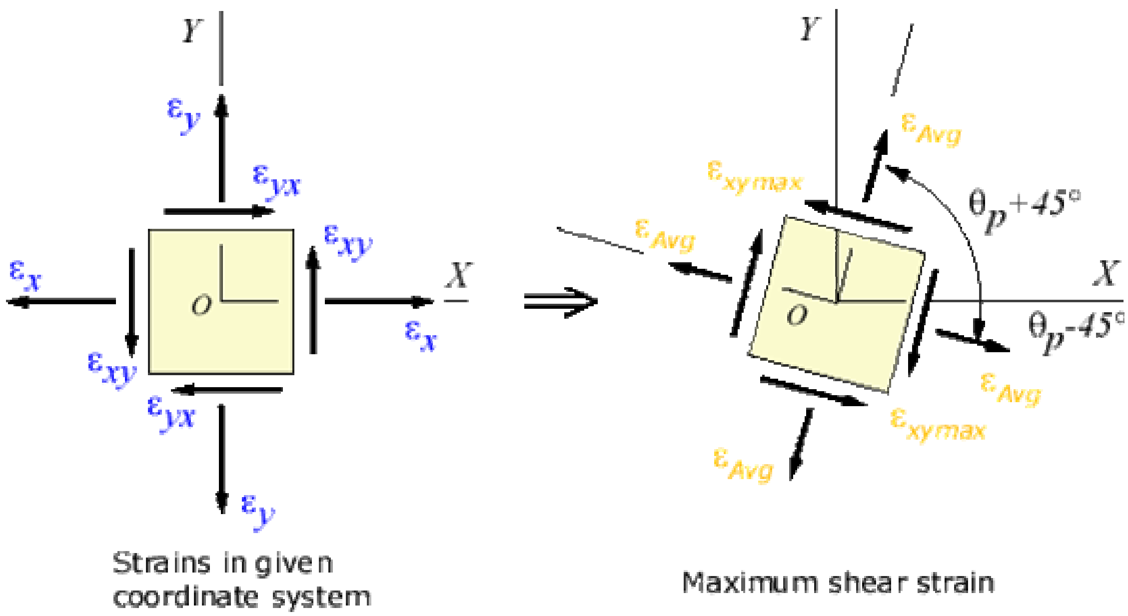
$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear strain is found to be one-half the difference between the two principal strains,

$$\varepsilon_{\max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

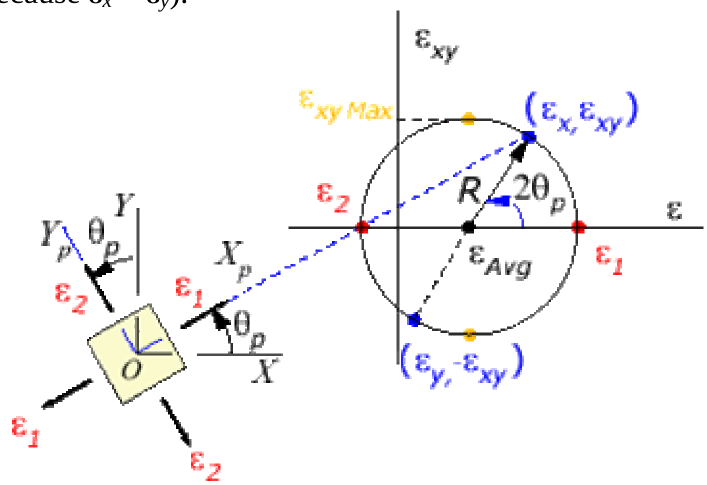
The transformation to the maximum shear strain direction can be illustrated as:



Solid Mechanics: Strain
Examples of Mohr's Circles in Plane Strain

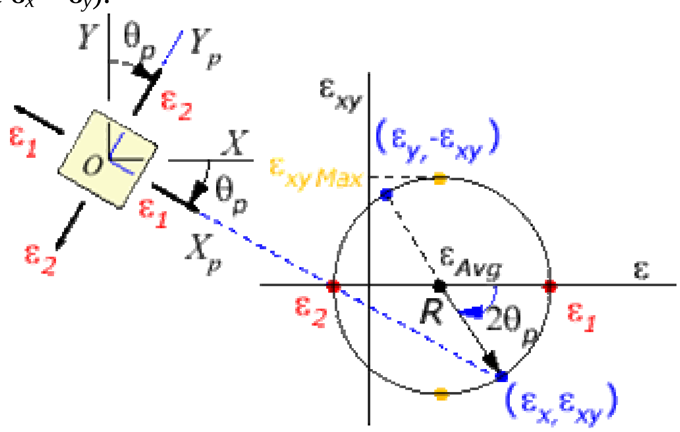
Case 1: $\varepsilon_{xy} > 0$ **and** $\varepsilon_x > \varepsilon_y$

The principal axes are counterclockwise to the current axes (because $\varepsilon_{xy} > 0$) and no more than 45º away (because $\varepsilon_x > \varepsilon_y$).



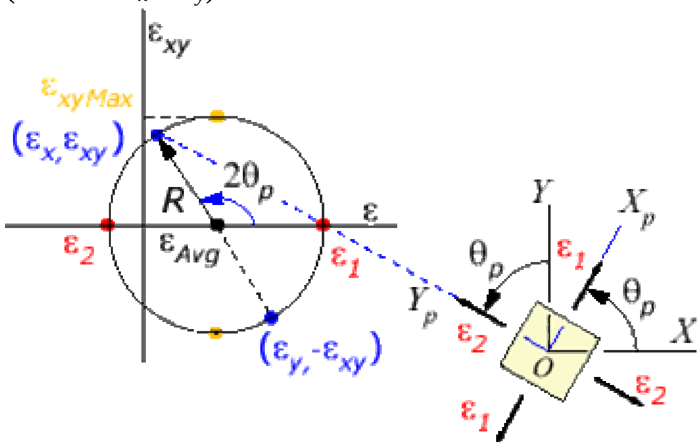
Case 2: $\varepsilon_{xy} < 0$ **and** $\varepsilon_x > \varepsilon_y$

The principal axes are clockwise to the current axes (because $\varepsilon_{xy} < 0$) and no more than 45º away (because $\varepsilon_x > \varepsilon_y$).



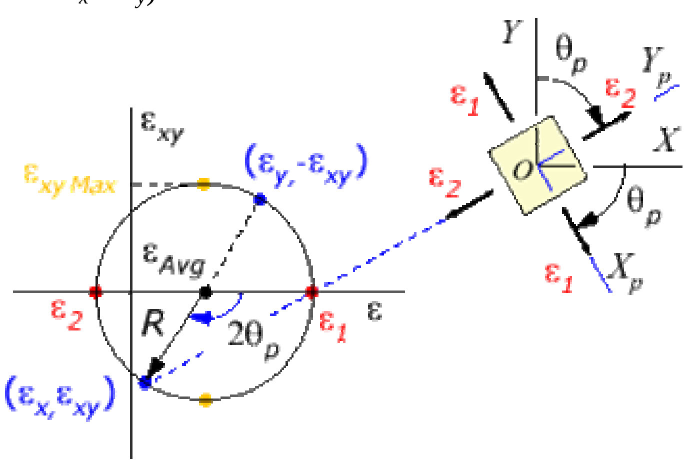
Case 3: $\varepsilon_{xy} > 0$ **and** $\varepsilon_x < \varepsilon_y$

The principal axes are counterclockwise to the current axes (because $\varepsilon_{xy} > 0$) and between 45º and 90º away (because $\varepsilon_x < \varepsilon_y$).



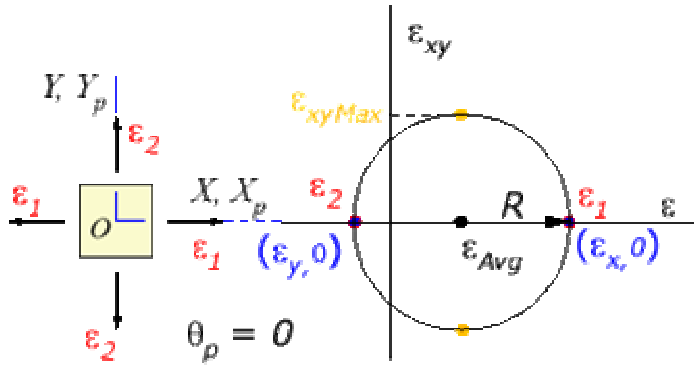
Case 4: $\varepsilon_{xy} < 0$ and $\varepsilon_x < \varepsilon_y$

The principal axes are clockwise to the current axes (because $\varepsilon_{xy} < 0$) and between 45º and 90º away (because $\varepsilon_x < \varepsilon_y$).



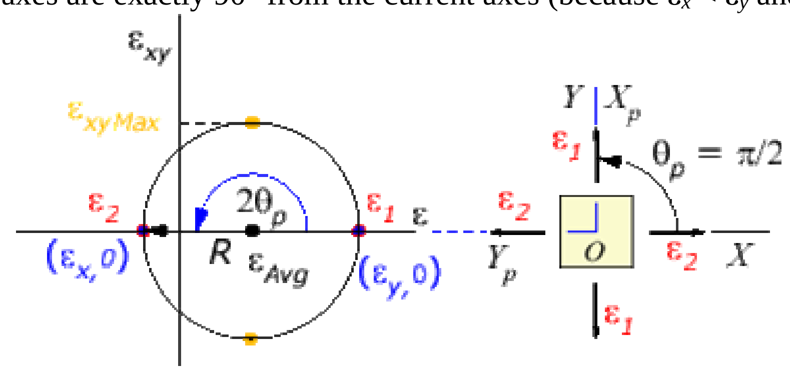
Case 5: $\varepsilon_{xy} = 0$ and $\varepsilon_x > \varepsilon_y$

The principal axes are aligned with the current axes (because $\varepsilon_x > \varepsilon_y$ and $\varepsilon_{xy} = 0$).



Case 6: $\varepsilon_{xy} = 0$ **and** $\varepsilon_x < \varepsilon_y$

The principal axes are exactly 90° from the current axes (because $\varepsilon_x < \varepsilon_y$ and $\varepsilon_{xy} = 0$).



Reference:

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